DUAL ABELIAN VARIETY IN CHARACTERISTIC 0

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Abstract. The aim of this talk is to construct the dual abelian variety in characteristic 0, which is a moduli space for translation invariant line bundles which we will therefore investigate. Moreover, we will see a few properties of duality, and in particular that Jacobians of curves are selfdual via the Θ-divisor coming from the Abel-Jacobi map in degree $g-1$.

Throughout this talk k is an algebraically closed field, A/k an abelian variety over k and C/k a smooth proper curve of genus $g > 0$. The assumption of characteristic 0 is only used in section 2.

In sections 1 and 2 we follow [Mum70], in section 3 [GMss] and in section 4 [Mil86b].

1. Pic^0 of an abelian variety

Definition 1.1. For a line bundle $\mathscr{L} \in \operatorname{Pic}(A)$ we consider

$$\phi_{\mathscr{L}} : A(k) \to \operatorname{Pic}(A), a \mapsto t_a^*\mathscr{L} \otimes \mathscr{L}^{-1},$$

and define $\operatorname{Pic}^0(A) := \{\mathscr{L} \in \operatorname{Pic}(A) : \phi_{\mathscr{L}} = 0\}$.

Recall that the theorem of the square asserts $t_{a+b}^*\mathscr{L} \otimes \mathscr{L} \cong t_a^*\mathscr{L} \otimes t_b^*\mathscr{L}$ for $\mathscr{L} \in \operatorname{Pic}(A)$ and $a, b \in A(k)$. Thus, $\phi_{\mathscr{L}}$ is a group homomorphism, and the image of $\phi_{\mathscr{L}}$ is contained in $\operatorname{Pic}^0(A)$. In particular, $\operatorname{Pic}^0(A) \subset \operatorname{Pic}(A)$ is a subgroup.

As we want to construct an abelian variety parametrizing $\operatorname{Pic}^0(A)$, we have to further investigate these line bundles. The following shows that Pic^0 is sensible notation, and that for an elliptic curve the moduli space we wish to construct is just the elliptic curve itself.

Lemma 1.2. *Let E/k be an elliptic curve with distinguished point $e \in E(k)$. Then*

$$\operatorname{Pic}^0(E) = \{\mathscr{L} \in \operatorname{Pic}(A) : \deg(\mathscr{L}) = 0\}.$$

Proof. For $x \in E(k)$ with $nx \neq e$ (recall that abelian varieties are divisible) we have

$$n\phi_{\mathscr{O}(e)}(-x) = \mathscr{O}(n[x] - n[e]) = \mathscr{O}(([nx] + (n-1)[e]) - n[e]) = \mathscr{O}([nx] - [e])$$

by the theorem of the square. But $\mathscr{O}([nx] - [e])$ is non-trivial as else $E \cong \mathbb{P}^1$ (see sheet 8, exercise 2 of our algebraic geometry II class). Now let $\mathscr{L} \in \operatorname{Pic}^0(A)$, $\mathscr{L} \cong \mathscr{O}(D)$ for a divisor $D = \sum_{x \in E(k)} n_x[x]$. Note that $\phi : \operatorname{Pic}(A) \to \operatorname{Hom}_{\mathrm{Grp}}(A(k), \operatorname{Pic}^0(A)), \mathscr{L} \mapsto \phi_{\mathscr{L}}$ is a group homomorphism, and $\phi_{\mathscr{L}} = \phi_{t_a^*\mathscr{L}}$ for all $a \in E(k)$ by the theorem of the square. Therefore, $\phi_{\mathscr{L}} = \deg(\mathscr{L})\phi_{\mathscr{O}(e)}$, which by the above is 0 if and only if $\deg(\mathscr{L}) = 0$. □

We can also express $\mathscr{L} \in \operatorname{Pic}^0(A)$ by triviality of a certain line bundle. To this end recall the following, which we saw in talk 11.

Proposition 1.3 ([Mil86a, Thm. 5.3])**.** *Let V/k be a proper variety and T/k a scheme of finite type, $\mathscr{L}$ a line bundle on $V \times T$. Then*

$$\{t \in T : \mathscr{L}\big|_{V \times \{t\}} \text{ trivial}\}$$

is closed.

Moreover, we recall the seesaw principle.

Theorem 1.4 (Seesaw principle, [Mil86a, Cor. 5.2])**.** *Let V/k be a proper variety and T/k an integral scheme of finite type over k. Moreover, let $\mathscr{L}, \mathscr{L}'$ be line bundles on $V \times T$. If $\mathscr{L}\big|_{V \times \{t\}} \cong \mathscr{L}'\big|_{V \times \{t\}}$ for all $t \in T(k)$ and $\mathscr{L}\big|_{\{v\} \times T} \cong \mathscr{L}'\big|_{\{v\} \times T}$ for some $v \in V(k)$, then $\mathscr{L} \cong \mathscr{L}'$.*

Lemma 1.5. *Let $\mathscr{L} \in \operatorname{Pic}(A)$. The following hold.*

(i) *$\mathscr{L} \in \operatorname{Pic}^0(A)$ if and only if $m^*\mathscr{L} \cong p_1^*\mathscr{L} \otimes p_2^*\mathscr{L}$ on $A \times A$.*
(ii) *For morphisms $f, g : S \to A$ of schemes and $\mathscr{L} \in \operatorname{Pic}^0(A)$ we have $(f+g)^*\mathscr{L} \cong f^*\mathscr{L} \otimes g^*\mathscr{L}$. In particular, $n_A^*\mathscr{L} \cong \mathscr{L}^n$.*
(iii) *If $\mathscr{L}$ is of finite order, then $\mathscr{L} \in \operatorname{Pic}^0(A)$.*
(iv) *We have $n_A^*\mathscr{L} \cong \mathscr{L}^{n^2} \otimes \mathscr{M}$ for some $\mathscr{M} \in \operatorname{Pic}^0(A)$.*

Proof. By the seesaw principle 1.4 $\mathscr{M} := m^*\mathscr{L} \otimes p_1^*\mathscr{L}^{-1} \otimes p_2^*\mathscr{L}^{-1}$ is trivial if and only if $\mathscr{M}\big|_{\{0\}\times A} \cong \mathscr{O}_A$ is trivial and for all $a \in A(k)$

$$\mathscr{M}\big|_{A\times\{a\}} \cong t_a^*\mathscr{L} \otimes \mathscr{L}^{-1}$$

is trivial. This yields (i).

For (ii) simply pull back this isomorphism along $(f,g) : S \to A \times A$ to obtain

$$(f+g)^*\mathscr{L} = (f,g)^*m^*\mathscr{L} = (f,g)^*(p_1^*\mathscr{L} \otimes p_2^*\mathscr{L}) = f^*\mathscr{L} \otimes g^*\mathscr{L}.$$

For (iii) note $\phi_{\mathscr{L}}(na) = n\phi_{\mathscr{L}}(a)$ for $a \in A(k)$ and use that abelian varieties are divisible.

It remains to show (iv). We already know

$$n_A^*\mathscr{L} \cong \mathscr{L}^{(n^2+n)/2} \otimes (-1)^*L^{(n^2-n)/2} = \mathscr{L}^{n^2} \otimes (\mathscr{L} \otimes (-1)^*\mathscr{L}^{-1})^{(n-n^2)/2},$$

so it suffices to show $\mathscr{L} \otimes (-1)^*\mathscr{L}^{-1} \in \mathrm{Pic}^0(A)$. For $a \in A(k)$

$$t_a^*(\mathscr{L} \otimes (-1)^*\mathscr{L}^{-1}) = t_a^*\mathscr{L} \otimes (-1)^*(\mathscr{L} \otimes t_{-a}^*\mathscr{L}^{-1}) \otimes (-1)^*\mathscr{L}^{-1}.$$

By 1.5 $(-1)^*(\mathscr{L} \otimes t_{-a}^*\mathscr{L}^{-1}) = \mathscr{L}^{-1} \otimes t_{-a}^*\mathscr{L}$, whence by the theorem of the square

$$t_a^*(\mathscr{L} \otimes (-1)^*\mathscr{L}) = t_a^*\mathscr{L} \otimes \mathscr{L}^{-1} \otimes t_{-a}^*\mathscr{L} \otimes (-1)^*\mathscr{L}^{-1} = \mathscr{L} \otimes (-1)^*\mathscr{L}^{-1},$$

as required. □

Lemma 1.6. *Let $\mathscr{L} \in \mathrm{Pic}(A)$. Then $n_A^*\mathscr{L} \cong \mathscr{L}^{n^2} \otimes \mathscr{M}$ for some $\mathscr{M} \in \mathrm{Pic}^0(A)$.*

Proof. We already know

$$n_A^*\mathscr{L} \cong \mathscr{L}^{(n^2+n)/2} \otimes (-1)^*L^{(n^2-n)/2} = \mathscr{L}^{n^2} \otimes (\mathscr{L} \otimes (-1)^*\mathscr{L}^{-1})^{(n-n^2)/2},$$

so it suffices to show $\mathscr{L} \otimes (-1)^*\mathscr{L}^{-1} \in \mathrm{Pic}^0(A)$. For $a \in A(k)$

$$t_a^*(\mathscr{L} \otimes (-1)^*\mathscr{L}^{-1}) = t_a^*\mathscr{L} \otimes (-1)^*(\mathscr{L} \otimes t_{-a}^*\mathscr{L}^{-1}) \otimes (-1)^*\mathscr{L}^{-1}.$$

By 1.5 $(-1)^*(\mathscr{L} \otimes t_{-a}^*\mathscr{L}^{-1}) = \mathscr{L}^{-1} \otimes t_{-a}^*\mathscr{L}$, whence by the theorem of the square

$$t_a^*(\mathscr{L} \otimes (-1)^*\mathscr{L}) = t_a^*\mathscr{L} \otimes \mathscr{L}^{-1} \otimes t_{-a}^*\mathscr{L} \otimes (-1)^*\mathscr{L}^{-1} = \mathscr{L} \otimes (-1)^*\mathscr{L}^{-1},$$

as required. □

Note that the following shows that for a family of line bundles on A parametrized by a variety it suffices to see that one of them lies in $\mathrm{Pic}^0(A)$ in order to see that it is a family of translation invariant line bundles.

Proposition 1.7. *Let S be a variety and $\mathscr{L}$ a line bundle on $A \times S$. Then for all $s_0, s_1 \in S(k)$*

$$\mathscr{L}_{s_0} \otimes \mathscr{L}_{s_1}^{-1} \in \mathrm{Pic}^0(A),$$

where $\mathscr{L}_s = \mathscr{L}\big|_{A\times\{s\}} \in \mathrm{Pic}(A_{k(s)})$ for $s \in S$.

Proof. Note that $p_2^*(\mathscr{L}^{-1}\big|_{\{0\}\times S})_s$ is trivial for all $s \in S(k)$, so by replacing $\mathscr{L}$ with $\mathscr{L} \otimes p_2^*(\mathscr{L}^{-1}\big|_{\{0\}\times S})$ we may assume that $\mathscr{L}\big|_{\{0\}\times S}$ is trivial. Moreover, by replacing $\mathscr{L}$ with $\mathscr{L} \otimes p_1^*\mathscr{L}_{s_0}^{-1}$ we may assume that $\mathscr{L}_{s_0}$ is trivial. We show that

$$\mathscr{M} := (m \times \mathrm{id})^*\mathscr{L} \otimes p_{13}^*\mathscr{L}^{-1} \otimes p_{23}^*\mathscr{L}^{-1} \text{ on } A \times A \times S$$

is trivial. Since $\mathscr{M}_s = m^*\mathscr{L}_s \otimes p_1^*\mathscr{L}_s^{-1} \otimes p_2^*L_s^{-1}$ for $s \in S(k)$ on $X \times X$ this suffices by 1.5. By commutativity of

$$\begin{array}{ccc} A \times \{0\} \times S & \longrightarrow & A \times A \times S \\ \downarrow & & \downarrow \\ \{0\} \times S & \longrightarrow & A \times S \end{array}$$

and triviality of $\mathscr{L}\big|_{\{0\}\times S}$ we have that $(p_{23}^*\mathscr{L}^{-1})\big|_{A\times\{0\}\times S}$ is trivial. This yields

$$\mathscr{M}\big|_{A\times\{0\}\times S} = \mathscr{L} \otimes \mathscr{L}^{-1},$$

triviality of $\mathscr{M}\big|_{\{0\}\times A\times S}$ follows similarly. Moreover,

$$\mathscr{M}\big|_{A\times A\times\{s_0\}} = m^*\mathscr{L}_{s_0} \otimes p_1^*\mathscr{L}_{s_0}^{-1} \otimes p_2^*\mathscr{L}_{s_0}^{-1}$$

is trivial by triviality of $\mathscr{L}_{s_0}$. The theorem of the cube yields the claim (talk 11). □

Proposition 1.8. *Let $\mathscr{L} \in \mathrm{Pic}^0(A)$ be non-trivial. Then $H^k(A, \mathscr{L}) = 0$ for all k.*

Proof. We proceed by induction on k. Suppose $H^0(A, \mathscr{L}) \neq 0$. Thus, since $\mathscr{L} \in \mathrm{Pic}^0(A)$,

$$\mathscr{L}^{-1} = (-1)^*\mathscr{L}$$

As multiplication by -1 is an automorphism, $\mathscr{L}^{-1}$ has global sections as well. But then $\mathscr{L} \cong \mathscr{O}_A$ since non-zero global sections $s \in H^0(A, \mathscr{L})$ and $s' \in H^0(A, \mathscr{L}^{-1})$ give morphisms $s : \mathscr{O}_A \to \mathscr{L}$, $s' : \mathscr{O}_A \to \mathscr{L}^{-1}$. Then

$s'^{\vee} \circ s : \mathscr{L} \to \mathscr{L}$ is non-zero and thus an isomorphism by $H^0(A, \mathscr{O}_A) = k$. But then $s : \mathscr{O}_A \to \mathscr{L}$ is already an isomorphism.

Now assume $H^i(A, \mathscr{L}) = 0$ for $i < k$ and some $k > 0$. Let $s : A \to A \times A, a \mapsto (a, 0)$. As $m \times s = \mathrm{id}_A$, $\mathrm{id} : H^k(A, \mathscr{L}) \to H^k(A, \mathscr{L})$ factors as

$$H^k(A, \mathscr{L}) \xrightarrow{m^*} H^k(A \times A, m^*\mathscr{L}) \xrightarrow{s^*} H^k(A, \mathscr{L}).$$

But $m^*\mathscr{L} = p_1^*\mathscr{L} \otimes p_2^*\mathscr{L}$ by 1.5, so Künneth yields

$$H^k(A \times A, m^*\mathscr{L}) = \sum_{i+j=k} H^i(A, \mathscr{L}) \otimes H^j(A, \mathscr{L}) = 0.$$

We conclude $H^k(A, \mathscr{L}) = 0$. □

The following is our main theorem, which asserts that the k-points of the abelian variety we wish to construct should be $A(k)/K(\mathscr{L})$ for ample $\mathscr{L}$, where $K(\mathscr{L}) = \{a \in A(k) : \phi_{\mathscr{L}}(a)\} = 0$. Recall from talk 11 that in this case $K(\mathscr{L})$ is finite. We need the following two results which are very similar to results we saw in talk 8.

Lemma 1.9 (Grauert,[Mum70, II.5, Cor. 2])**.** *Let $f : X \to Y$ be a proper morphism of Noetherian schemes with Y reduced and connected, and let $\mathscr{F}$ be a coherent sheaf on X which is flat over Y. Then, if $y \in Y \mapsto \dim_{k(y)} H^p(X_y, \mathscr{F}_y)$ is constant, $R^{p-1}f_*\mathscr{F} \otimes_{\mathscr{O}_Y} k(y) \cong H^{p-1}(X_y, \mathscr{F}_y)$ for all $y \in Y$.*

With this, one also has the following.

Corollary 1.10 (Grauert,[Mum70, II.5, Cor. 4])**.** *In the situation of 1.9 if $R^k f_*\mathscr{F} = 0$ for $k \geq k_0$, then $H^k(X_y, \mathscr{F}_y) = 0$ for all $y \in Y$ and $k \geq k_0$.*

From now on fix an ample $\mathscr{L} \in \mathrm{Pic}(A)$.

Theorem 1.11. *The group morphism*

$$\phi_{\mathscr{L}} : A(k) \to \mathrm{Pic}^0(A)$$

is surjective.

Proof. Let $\mathscr{M} \in \mathrm{Pic}^0(A)$ and assume $t_a^*\mathscr{L} \otimes \mathscr{L}^{-1} \not\cong \mathscr{M}$ for all $a \in A(k)$. On $A \times A$ consider the line bundle

$$\mathscr{K} := m^*\mathscr{L} \otimes p_1^*\mathscr{L}^{-1} \otimes p_2^*(\mathscr{L}^{-1} \otimes \mathscr{M}^{-1}).$$

Note that for $a \in A(k)$

$$\mathscr{K}\big|_{\{a\}\times A} \cong t_a^*\mathscr{L} \otimes \mathscr{L}^{-1} \otimes \mathscr{M}^{-1},$$

which lies in $\mathrm{Pic}^0(A)$ and by assumption is non-trivial. Thus, by 1.8 $H^i(A, \mathscr{K}\big|_{\{a\}\times A}) = 0$ for all i, and Grauert's lemma 1.9 (together with Nakayama) yields $R^i p_{1,*}\mathscr{K} = 0$. We deduce

$$R\Gamma(A, Rp_{1,*}\mathscr{K}) = R\Gamma(A \times A, \mathscr{K}) = 0.$$

Moreover, $t_a^*\mathscr{L} \otimes \mathscr{L}^{-1} \cong \mathscr{K}\big|_{A\times\{a\}}$ is non-trivial for $a \in A(k) \setminus K(L)$ and $K(L)$ is closed, whence as above by 1.9 $\mathrm{supp}(R^i p_{2,*}\mathscr{K}) \subset K(\mathscr{L})$ (for this, note that the above implies that the higher direct image sheaves are 0 if we restrict to $p_2^{-1}(A \setminus K(\mathscr{L}))$ and use that restriction is exact). But $K(\mathscr{L})$ is finite, so

$$Rp_{2,*}\mathscr{K} \cong \iota_*\iota^{-1}Rp_{2,*}\mathscr{K}$$

and $\iota_* = R\iota_*$ as well as $R\Gamma(K(\mathscr{L}), -) = \Gamma(K(\mathscr{L}), -)$, where $\iota : K(\mathscr{L}) \to A$. Therefore,

$$0 = R\Gamma(A, Rp_{2,*}\mathscr{K}) = R\Gamma(K(\mathscr{L}), \iota^{-1}Rp_{2,*}) = \Gamma((K(\mathscr{L}), \iota^{-1}Rp_{2,*}) = \bigoplus_{x \in K(\mathscr{L})} (Rp_{2,*})_x.$$

We deduce $Rp_{2,*}\mathscr{K} = 0$, and 1.10 yields $H^i(A, \mathscr{K}\big|_{A\times\{a\}}) = 0$ for all $a \in A$. But $\mathscr{K}\big|_{A\times\{0\}}$ is trivial, and thus has global sections; a contradiction. □

2. The dual abelian variety in characteristic 0

Theorem 1.11 suggests that our moduli space of Pic^0 on k-points should be $A(k)/K(\mathscr{L})$. As we hope for an abelian variety, $K(\mathscr{L})$ ought to be the k-points of a closed group subscheme of A. But in characteristic 0 group schemes are smooth by a theorem of Cartier [Sta22, Tag 047N], in particular reduced, and we have our natural candidate: $K(\mathscr{L})$ with its reduced subscheme structure.

We remark that in arbitrary characteristic this does not work, but one has to see that there is a closed group subscheme $K(\mathscr{L})$ with functor of points $K(\mathscr{L})(S) = \{a \in A(S) : t_a^*\mathscr{L}_S \cong \mathscr{L}_S \text{ on } X \times S\}$.

Definition 2.1. An abelian variety $A^\vee$ together with a line bundle $\mathscr{P}$ on $A \times A^\vee$ called *Poincaré bundle* is the *dual abelian variety of* A if

(1) $\mathscr{P}\big|_{\{0\}\times A^\vee} \cong \mathscr{O}_{A^\vee}$ and $\mathscr{P}\big|_{A\times\{a\}} \in \mathrm{Pic}^0(A_{k(A)})$ for all $a \in A^\vee$, and

(2) for all schemes T/k and line bundles $\mathscr{M}$ on $A \times T$ with $\mathscr{M}\big|_{\{0\}\times T} \cong \mathscr{O}_T$ and $\mathscr{M}_{A\times\{t\}} \in \mathrm{Pic}^0(A_{k(t)})$ for all $t \in T$ there is a unique morphism $f : T \to A^\vee$ s.t. $(1 \times f)^*\mathscr{P} \cong \mathscr{M}$.

In other words, $A^\vee$ is an abelian variety representing the functor

$$(\mathrm{Sch}/k)^{\mathrm{opp}} \mapsto \mathrm{Sets}, T \mapsto \{\mathscr{M} \in \mathrm{Pic}(A \times T) : \mathscr{M}\big|_{\{0\}\times T} \cong \mathscr{O}_T,\ \mathscr{M}\big|_{A\times\{t\}} \in \mathrm{Pic}^0(A_{k(t)}) \text{ for all } t \in T\}.$$

Note that for an elliptic curve E/k this functor is simply $\mathrm{Pic}^0_{E/k,0}$.

As noted above in characteristic 0 we already have a candidate for $A^\vee$, and it remains to to construct the Poincaré bundle. For the rest of this section assume $\mathrm{char}(k) = 0$ and put $A^\vee := A/K(\mathscr{L})$, where $K(\mathscr{L})$ has the reduced subscheme structure.

In order to obtain the Poincaré bundle we use descent of line bundles as discussed in talk 13.

Theorem 2.2. *Let G be a finite group scheme acting freely on A, i.e., $G \times A \to A \times A, (g, a) \mapsto (g.a, a)$ is a closed immersion. Denote the natural map $X \to X/G$ by π. Then*

$$\mathrm{QCoh(X/G)} \to \{\mathscr{F} \in \mathrm{QCoh(X)}\ G-\text{equivariant}\}, \mathscr{F} \mapsto \pi^*\mathscr{F}$$

is an equivalence of categories under which locally free sheaves of some rank correspond to locally free sheaves of the same rank.

Recall that a G-equivariant sheaf on A is simply a sheaf $\mathscr{F}$ on A together with isomorphisms $\lambda_g : g^*\mathscr{F} \to \mathscr{F}$ for all $g \in G$ s.t. for all $g, h \in G$

$$(g+h)^*\mathscr{F} \xrightarrow{\lambda_{g+h}} \mathscr{F}, \qquad (g+h)^*\mathscr{F} \xrightarrow{g^*\lambda_g} g^*\mathscr{F} \xrightarrow{\lambda_h} \mathscr{F}$$

commutes.

We want $\pi = \phi_{\mathscr{L}}$, which thus has to correspond to $\Lambda := m^*\mathscr{L} \otimes p_1^*\mathscr{L} \otimes p_2^*\mathscr{L}^{-1}$ as $\Lambda\big|_{\{a\}\times A} = t_a^*\mathscr{L} \otimes \mathscr{L}^{-1}$ for all $a \in A(k)$. Hence, we have to give Λ on $A \times A$ a $\{0\} \times K(\mathscr{L})$-equivariant structure in order to obtain a line bundle $\mathscr{P}$ on $A \times A^\vee = (A \times A)/(\{0\} \times K(\mathscr{L}))$ with $\pi^*\mathscr{P} = \Lambda$.

Lemma 2.3. *There exists an $\{0\} \times K(\mathscr{L})$-equivariant structure on Λ.*

Proof. First, note that

$$t^*_{(0,a)}\Lambda = t^*_{(0,a)}\Lambda \cong m^*t_a^*\mathscr{L} \otimes p_1^*\mathscr{L}^{-1} \otimes p_2^*t_a^*\mathscr{L}^{-1} \cong \Lambda$$

for all $a \in K(\mathscr{L})$. Thus, isomorphisms $\lambda_a : t^*_{(0,a)}\Lambda \cong \Lambda$ for $a \in K(\mathscr{L})$ exist, but we need

(1)
$$t^*_{(0,a+b)}\Lambda \xrightarrow{\lambda_{a+b}} \Lambda, \qquad t^*_{(0,a+b)}\Lambda \xrightarrow{t^*_{(0,a)}\lambda_h} t^*_{(0,a)}\Lambda \xrightarrow{\lambda_a} \Lambda$$

to commute for $a, b \in K(\mathscr{L})$. Fix an isomorphism $\Lambda\big|_{\{0\}\times A} \cong \mathscr{O}_A$. As $\mathrm{res} : k^\times = H^0(A \times A, \mathscr{O}^\times_{A\times A}) \xrightarrow{\sim} H^0(\{0\} \times A, \mathscr{O}^\times_{\{0\}\times A}) = k^\times$, we can simply require the λ_a to be the unique isomorphisms $t^*_{(0,a)}\Lambda \xrightarrow{\sim} \Lambda$ which after restriction to $\{0\} \times A$ are

$$t_a^* : t_a^*\mathscr{O}_A \cong t_a^*\mathscr{L}\big|_{\{0\}\times A} = (t^*_{(0,a)}\mathscr{L})\big|_{\{0\}\times A} \xrightarrow{\sim} \mathscr{L}\big|_{\{0\}\times A} \cong \mathscr{O}_A.$$

Since $t_a \circ t_b = t_{a+b}$ the commutativity of 1 after restriction to $\{0\} \times A$ is clear, and we obtain a line bundle $\mathscr{P}$ on $A \times A^\vee$ with $\pi^*\mathscr{P} \cong \Lambda$. □

Theorem 2.4. *The dual abelian variety of A is $A^\vee$ and $\mathscr{P}$ is the Poincaré bundle.*

Proof. We outline the proof only for normal varieties S/k, for the general case including arbitrary characteristic see [Mum70, III.13].

Let $\mathscr{M}$ be a line bundle on $A \times S$ s.t. $\mathscr{M}\big|_{\{0\}\times S}$ is trivial and $\mathscr{M}\big|_{A\times\{s\}} \in \mathrm{Pic}^0(A_{k(s)})$ for all $s \in S$. On $A \times S \times A^\vee$ consider

$$\mathscr{F} := p_{12}^*\mathscr{M} \otimes p_{13}^*\mathscr{P}^{-1}.$$

Note that

$$\mathscr{F}\big|_{A\times\{(s,b)\}} \cong \mathscr{M}_s \otimes \mathscr{P}_b^{-1}$$

for $s \in S(k), b \in A^\vee$. Moreover, consider

$$\Gamma := \{(s, b) \in S \times A^\vee : \mathscr{F}\big|_{A\times\{(s,b)\}} \text{ trivial}\},$$

which is closed in $S \times A^\vee$ by 1.4 (of course we equip Γ with its reduced subscheme structure). For $(s, b) \in \Gamma(k)$ we have $\mathscr{M}_s \cong \mathscr{P}_b$. By construction of $A^\vee$ we have $A^\vee(k) = A(k)/K(\mathscr{L})$, so for every $s \in S(k)$ there is a unique $b \in A^\vee(k)$ with $\mathscr{M}_s \cong \mathscr{P}_b$. Note that this in particular implies that the morphism f we will obtain is unique with $(1 \times f)^*\mathscr{P} \cong \mathscr{M}$. We conclude that on k-points the projection

$$p_1 : \Gamma \to S, (s, b) \mapsto s$$

is a bijection. Therefore, $k(A^\vee)/k(\Gamma)$ is a field extension of separable degree 1, and by $\mathrm{char}(k) = 0$ we even get $A^\vee(k) = \Gamma(k)$ [Sha13, p. 142, Thm. 2.29]. The following theorem shows that $p_1 : \Gamma \to S$ is even an isomorphism, and $f : S \cong \Gamma \xrightarrow{p_2} A^\vee$ fulfils $\mathscr{M} \cong (1 \times f)\mathscr{P}$ by the seesaw principle 1.4 as both

$$\mathscr{M}\big|_{\{0\}\times S} \cong \mathscr{O}_s \cong f^*(\mathscr{P}\big|_{\{0\}\times A^\vee}) \cong ((1 \times f)^*\mathscr{P})\big|_{\{0\}\times S}$$

by triviality of $\mathscr{P}\big|_{\{0\}\times A^\vee}$ and and for all $s \in S(k)$

$$\mathscr{M}_s \cong \mathscr{P}_{f(s)} \cong ((1 \times f)^*\mathscr{P})\big|_{A\times\{f(s)\}}.$$

□

Theorem 2.5 (Zariski's main theorem, [Liu02, 4, Cor. 4.6])**.** *Let $f : X \to S$ be a quasi-finite and birational morphism of varieties with S normal. Then f is an open immersion.*

3. Dual morphisms

We now show that duality of abelian varieties is a "good" duality in the sense that $A \cong A^{\vee\vee}$. For this, we need Cartier duals, which are the scheme version of character groups.

Theorem 3.1 ([Mum70, III.14])**.** *Let G/k be a finite group scheme. Then there is a finite group scheme G^D of the same rank as G which represents*

$$(\mathrm{Sch}/k)^{\mathrm{opp}} \mapsto \mathrm{Sets}, T \mapsto \mathrm{Hom}_{\mathrm{GrpSch}/T}(G \times T, \mathbb{G}_m \times T).$$

With these, the following important theorem holds.

Theorem 3.2 ([Mum70, III.15, Thm. 1])**.** *Let $f : A \to B$ be an isogeny. Then $f^\vee : B^\vee \to A^\vee$ is an isogeny and naturally $\ker(f^\vee) \cong \ker(f)^D$.*

Note that this is just the scheme version of the duality between $\ker(f^* : \mathrm{Pic}(B) \to \mathrm{Pic}(A))$ and $\ker(f)$ as finite abelian groups we saw before in talk 13.

Corollary 3.3. *Let $f : A \to B$ be a homomorphism of abelian varieties and $\mathscr{M}$ a line bundle on B, $\mathscr{N} := f^*\mathscr{M}$. Then $\phi_{\mathscr{N}} : A \to A^\vee$ factors as*

$$A \xrightarrow{f} B \xrightarrow{\phi_{\mathscr{M}}} B^\vee \xrightarrow{f^\vee} B^\vee.$$

Thus, if f is an isogeny and $\mathscr{M}$ ample, then $\mathscr{N}$ is ample and $\mathrm{rank}(K(\mathscr{N})) = \mathrm{rank}(K(\mathscr{M})) \cdot \deg(f)^2$.

Proof. Just use

$$t_a^* f^* \mathscr{M} \cong f^* t_{f(a)}^* \mathscr{M}$$

for all $a \in A$, whence

$$\phi_{\mathscr{N}}(a) = f^*(t_{f(a)}^*\mathscr{M} \otimes \mathscr{M}^{-1}) = f^\vee(\phi_{\mathscr{M}}(f(a)))$$

for $a \in A(k)$. Using the theorem 3.3 and that $\phi_{\mathscr{M}}$ and $\phi_{\mathscr{N}}$ are isogenies if and only if $\mathscr{M}$ and $\mathscr{N}$ are ample respectively the second part is clear. □

As the Poincaré bundle $\mathscr{P}$ is a line bundle on $A \times A^\vee \cong A^\vee \times A$ we also obtain a morphism $\iota_A : A \to A^{\vee\vee}$.

Lemma 3.4. *Let $\mathscr{L}$ be a line bundle on A. Then $\phi_{\mathscr{L}} = \phi_{\mathscr{L}}^\vee \circ \iota_A$.*

Proof. This is purely formal. Put $\Lambda := m^*\mathscr{L} \otimes p_1^*\mathscr{L}^{-1} \otimes p_2^*\mathscr{L}^{-1}$ on $A \times A$, which is the line bundle representing $\phi_{\mathscr{L}}$. Also, let $s : A \times A \to A \times A$ be the swap (by abuse of notation we also write this for the swap $A \times A^\vee \cong A^\vee \times A$). We write $[\mathscr{N}]$ for the morphism corresponding to a line bundle $\mathscr{N}$ on $A \times A^\vee$. Then, as $s^*\Lambda \cong \Lambda$,

$$\phi_{\mathscr{L}} = [\Lambda] = [s^*\Lambda] = [s^*(1 \times \phi_{\mathscr{L}})^*\mathscr{P}] = [(\phi_{\mathscr{L}} \times 1)^* s^*\mathscr{P}] = \phi_{\mathscr{L}}^\vee \circ \iota_A.$$

□

Theorem 3.5. *The morphism $\iota_A : A \to A^\vee$ is an isomorphism.*

Proof. Since $\phi_{\mathscr{L}} = \phi_{\mathscr{L}}^\vee \circ \iota_A$ also for ample $\mathscr{L}$, $\ker(\iota_A)$ is finite and thus an isogeny as $\dim(A) = \dim(A^\vee)$. Moreover,

$$\deg(\phi_{\mathscr{L}}) = \deg(\phi_{\mathscr{L}}^\vee) \cdot \deg(\iota_A),$$

so 3.1 shows that $\deg(\iota_A) = 1$, and ι_A is an isomorphism. □

Definition 3.6. A homomorphism $f : A \to A^\vee$ is called *symmetric* if $f = f^\vee \circ \iota_A$. If $f = \phi_{\mathscr{L}}$ for some ample $\mathscr{L} \in \mathrm{Pic}(A)$, then f is an isogeny, and f is called a *polarization of degree* $\deg(f)$. If a polarization f is even an isomorphism, it is called a *principal polarization.*

4. Jacobians are principally polarized via the Θ-divisor

Let $p \in C(k)$ be a k-rational point of C, and $\mathscr{L}^p_{\text{univ}} \in \text{Pic}^0_{C/k,p}(\underline{\text{Pic}}^0_{C/k})$ be the universal object of $\text{Pic}^0_{C/k,p}$, cf. talks 6 and 7. We define $J \coloneqq \underline{\text{Pic}}^0_{C/k}$ to be the Jacobian of C. Moreover, we denote the Abel-Jacobi map by AJ, and obtain maps

$$f^{(d)} : C^{(d)} = \underline{\text{Hilb}}^d_{C/k} \xrightarrow{\text{AJ}} \underline{\text{Pic}}^d_{C/k} \xrightarrow{-d[p]} J,$$

which are proper as both $\underline{\text{Hilb}}^d_{C/k}, J$ are proper over $\text{Spec}(k)$. In particular,

$$W^d \coloneqq f^{(d)}(C^{(d)}) \subset J$$

is closed. As $\dim(C^{(d)}) = d$, $\dim(J) = g$ and $f^{(g)}$ is surjective we have that W^d is of codimension $g - d$ for $1 \leq d \leq g$. As $C^{(d)}$ is also irreducible, we conclude that

$$\Theta \coloneqq W^{g-1}$$

is irreducible. In particular, Θ is a divisor. For $a \in J(k)$ we put

$$\Theta^{-1} \coloneqq (-1) \cdot \Theta, \quad \Theta_a \coloneqq \Theta + a, \quad \Theta^-_a \coloneqq \Theta^- + a.$$

The following now holds.

Lemma 4.1. *There is a non-empty open $U \subset J$ s.t.*

(1) the fibers of $f^{(g)}$ at any $u \in U$ are 0-dimensional, and

(2) if $a \in U(k)$ and $D(a) \in C^{(g)}(k)$ with $f^{(g)}(D(a)) = a$ (i.e., a is degree 0 line bundle and $D(a)$ a divisor of degree g s.t. $a \cong \mathscr{O}(D - g[p])$), then $D(a) = [p_1] + \ldots + [p_g]$ for unique and distinct $p_1, \ldots, p_g \in C(k)$.

Moreover, for $a \in U(k)$ we have $f^{-1}\Theta^-_a = D(a)$ (as divisors), where $f = f^{(1)} : C \to J$.

Proof. As $\dim(C^{(g)}) = \dim(J)$ and $f^{(g)}$ is onto, there is a nonempty open of J s.t. (1) holds, see [Sha13, p. 75, Thm. 1.25]. For the second part one simply also takes out images of subschemes of the form $\Delta \times C^{g-2} \subset C^g$, where Δ is the diagonal, which are closed in J as $C^g \to C^{(g)} \xrightarrow{f^{(g)}} J$ is proper.

Let $a \in U(k)$, $D(a) = \sum_{i=1}^g [p_i]$. A point $x \in C(k)$ gets mapped to Θ^-_a by f if and only if there are $q_2, \ldots, q_g \in C(k)$ with $f(x) = -f(q_2) - \ldots - f(q_g) + a$. This implies $f^{(g)}([x] + [q_2] + \ldots + [q_g]) = a$, whence by construction of U we must have $x \in \{p_1, \ldots, p_g\}$, and thus $f^{-1}(\Theta^-_a) = n_1[p_1] + \ldots + n_g[p_g]$ for some $n_1, \ldots, n_g \geq 0$. Therefore, it suffices to show $\deg(f^{-1}(\Theta^-_a)) = g$. For this see [Mil86b, Lemma 6.7]. ☐

Fix an open $U \subset J$ as in the lemma.

Corollary 4.2. *The following hold.*

(1) Let $a \in J(k)$ and $f^{(g)}(D(a)) = a$ for a divisor $D(a) \in C^{(g)}$, then $f^*\mathscr{O}(\Theta^-_a) \cong \mathscr{O}(D(a))$.

(2) On $C \times J$ we have

$$(f \times (-1))^*\Lambda(\Theta^-) \cong \mathscr{L}^p_{\text{univ}},$$

where $\Lambda(\Theta^{-1}) \coloneqq \Lambda(\mathscr{O}(\Theta^-)) \coloneqq m^*\mathscr{O}(\Theta^-) \otimes p_1^*\mathscr{O}(\Theta^-)^{-1} \otimes p_2^*\mathscr{O}(\Theta^-)^{-1}$ on $J \times J$.

Proof. By 4.2 part 1 holds for $u \in U(k)$. Moreover, for $a \in U(k)$ we have

$$(f \times (-1))^* m^*\mathscr{O}(\Theta^-)\big|_{C \times \{a\}} = f^* t^*_{-a}\mathscr{O}(\Theta^-) = f^*\mathscr{O}(\Theta^-_a)$$

as well as

$$\mathscr{L}^p_{\text{univ}}\big|_{C \times \{a\}} \cong a \cong \mathscr{O}(D(a) - g[p]).$$

Thus, it suffices to show that

$$(f \times (-1))^* m^*\mathscr{O}(\Theta^-)^{-1} \otimes \mathscr{L}^p_{\text{univ}} \otimes p_1^*\mathscr{O}(g[p])$$

is trivial if restricted to $C \times \{a\}$ for all $a \in C(k)$. Since this already holds for $a \in U(k)$, this follows from 1.3.

If we put $a = 0$, we see $f^*\mathscr{O}(\Theta^-) \cong \mathscr{O}(g[p])$. Since

$$(f \times (-1))^* p_1^*\mathscr{O}(\Theta^-)\big|_{C \times \{a\}} = f^*\mathscr{O}(\Theta^-)$$

we conclude

$$\mathscr{K} \coloneqq (f \times (-1))^*(m^*\mathscr{O}(\Theta^-)^{-1} \otimes p_1^*\mathscr{O}(\Theta^-) \otimes p_2^*\mathscr{O}(\Theta^-)) \otimes \mathscr{L}^p_{\text{univ}}$$

is still trivial if restricted to $C \times \{a\}$. As $\mathscr{L}^p_{\text{univ}}\big|_{\{p\} \times J}$ is trivial and $f(p) = 0$ also $\mathscr{K}\big|_{\{p\} \times J}$ is trivial. The Seesaw principle 1.4 yields part 2. ☐

With this preparation we are now ready to proof that J is principally polarized via the Θ-divisor. Note that $f(p) = 0$ yields $(f \times \text{id})^*\mathscr{P}\big|_{\{p\} \times J^\vee} \cong \mathscr{P}\big|_{\{0\} \times J^\vee}$, and

$$((f \times \text{id})^*\mathscr{P})\big|_{C \times \{0\}} \cong f^*(\mathscr{P}\big|_{J \times \{0\}}) \cong f^*\mathscr{O}_J \cong \mathscr{O}_C.$$

Since $j \in J^\vee \mapsto \deg(C_{k(j)}, ((f \times \text{id})^*\mathscr{P})\big|_{C \times \{j\}})$ is locally constant (cf. talk 7) $(f \times \text{id})^*\mathscr{P}$ represents a function $\varphi : J^\vee \to J$. Here, $\mathscr{P}$ is the Poincare bundle of J on $J \times J^\vee$.

Theorem 4.3. *The functions* $-\varphi$ *and* $\phi_{\mathscr{O}(\Theta)}$ *are inverses.*

Proof. Note that by 1.5 we have $\phi_{\mathscr{O}(\Theta)} = \phi_{(-1)^*\mathscr{O}(\Theta)} = \phi_{\mathscr{O}(\Theta^-)}$. Moreover, as $(f \times \mathrm{id})^*\mathscr{P}$ and $(1 \times \varphi)^*\mathscr{L}^p_{\mathrm{univ}}$ both represent φ, these are isomorphic. We have

$$\begin{aligned}(1 \times -\phi_{\mathscr{O}(\Theta)})^*(1 \times \varphi)^*\mathscr{L}^p_{\mathrm{univ}} =& (1 \times -\phi_{\mathscr{O}(\Theta)})^*(f \times \mathrm{id})^*\mathscr{P} = (f \times (-1))^*(1 \times \phi_{\mathscr{O}(\Theta^-)})^*\mathscr{P} \\ =& (f \times (-1))^*\Lambda(\Theta^-) \cong \mathscr{L}^p_{\mathrm{univ}}\end{aligned}$$

by 4.2 as $\Lambda(\Theta^-) = m^*\mathscr{O}(\Theta) \otimes p_1^*\mathscr{O}(\Theta)^{-1} \otimes p_2^*\mathscr{O}(\Theta)^{-1}$ represents $\phi_{\mathscr{O}(\Theta^-)}$.

Thus, $\varphi \circ (-\phi_{\mathscr{O}(\Theta)}) = \mathrm{id}_J$. In particular, $\phi_{\mathscr{O}(\Theta)}$ has trivial kernel whence $\mathscr{O}(\Theta)$ is ample. Therefore, $\phi_{\mathscr{O}(\Theta)}$ is an isogeny by 1.11 (or $\dim(J) = \dim(J^\vee)$). We conclude that $\phi_{\mathscr{O}(\Theta)}$ is an isomorphism. As $\phi_{\mathscr{O}(\Theta)}$ is a group morphism, $\varphi \circ (-\phi_{\mathscr{O}(\Theta)}) = \mathrm{id}_J$ shows $\varphi \circ (-1) = -\varphi$, so indeed $-\varphi$ is inverse to $\phi_{\mathscr{O}(\Theta)}$. □

It should be mentioned that choosing another k-rational point $p' \in C(k)$ in place of p simply results in a translation of the Θ-divisor by $\mathscr{O}(g[p'] - g[p])$, but this does not change the polarization as clearly $\phi_{\mathscr{L}} = \phi_{t_a^*\mathscr{L}}$ for all $a \in J(k)$ and line bundles $\mathscr{L}$ on J.

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